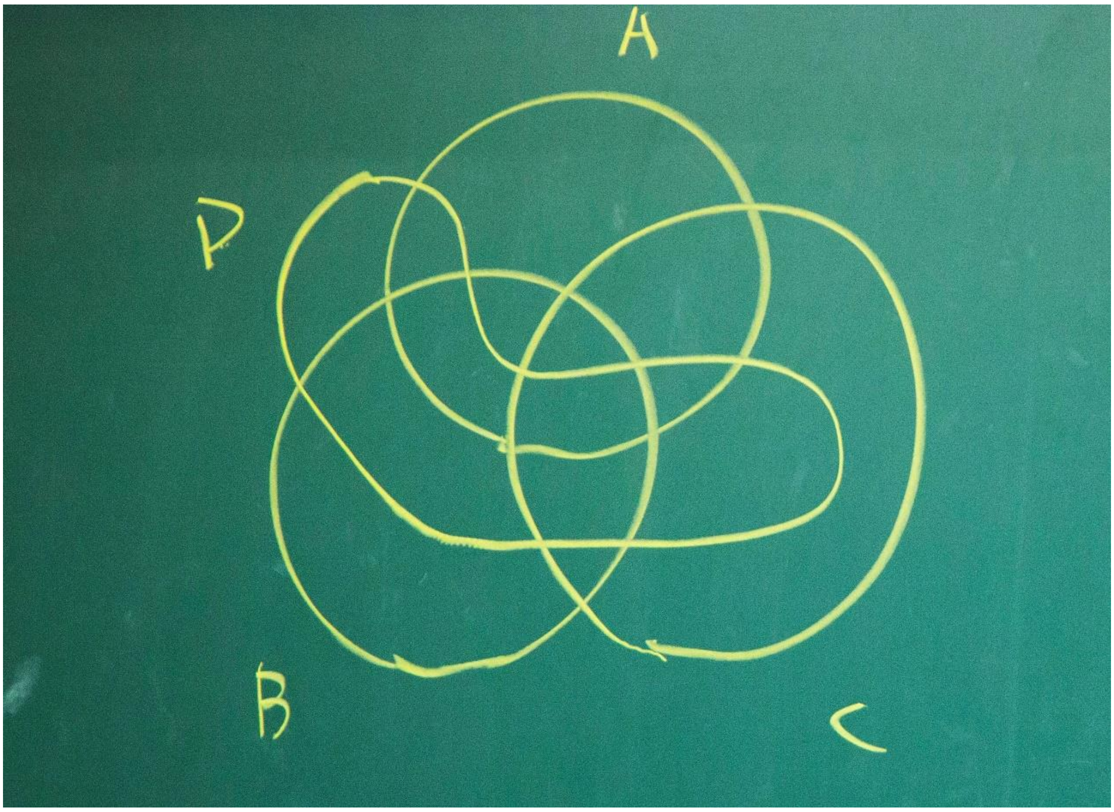
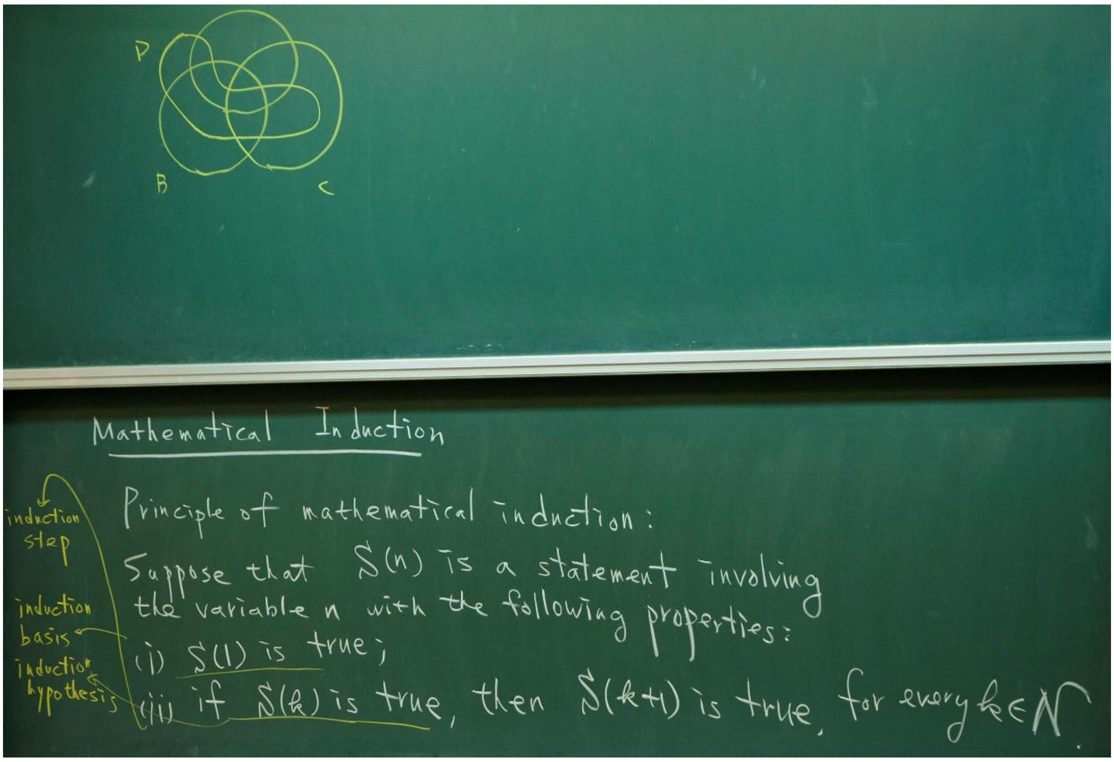
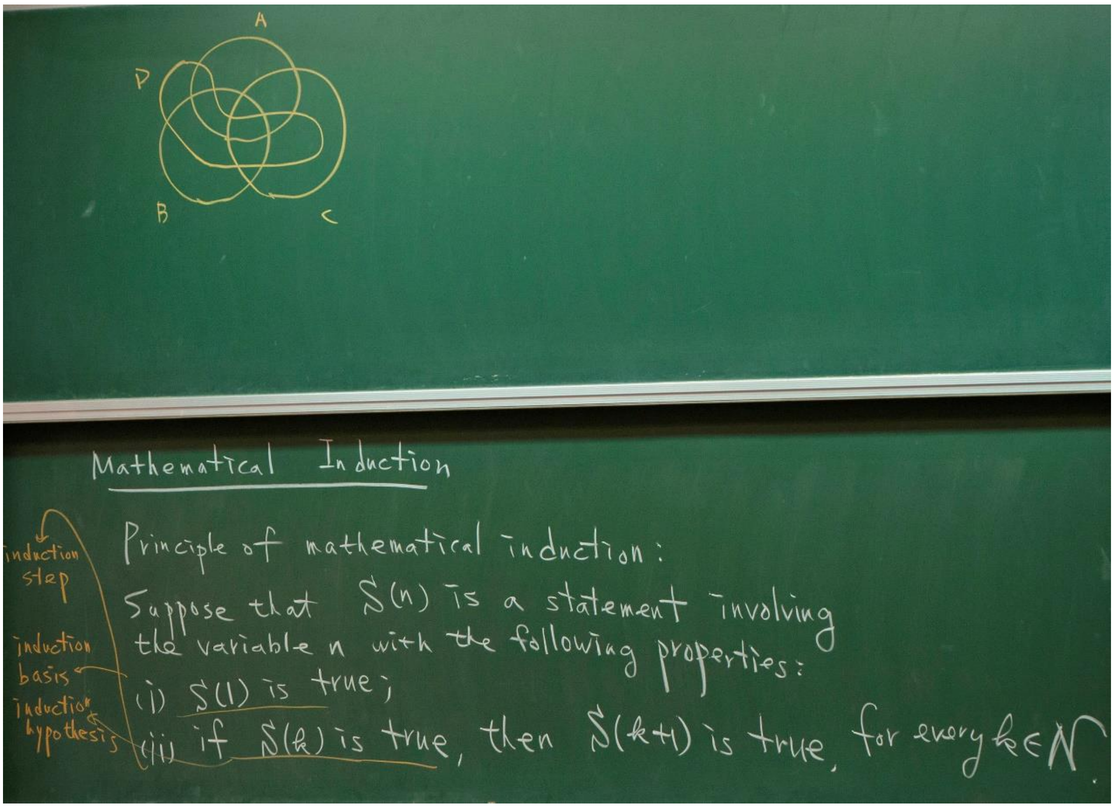
【10920 趙啟超教授離散數學 / 第 3 堂版書】




P
B
C
Mathematical Induction
Principle of mathematical induction:
Suppose that S(n) is a statement involving the variable n with the following properties:
(i) S(1) is true;
(ii) if S(k) is true, then S(k+1) is true, for every k∈N.
induction step
induction basis
induction hypothesis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Then $S(n)$ is true for all $n \in \mathbb{N}$.

<u>Example</u> Show that for all $n \in \mathbb{N}$.

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<u>Proof</u> Induction basis: When $n=1$,

$$\sum_{i=1}^{1} (4i+3) = 7 = 2\cdot 1^2+5\cdot 1$$

Induction step: Assume that $\sum_{i=1}^{k} (4i+3) = 2k^2+5k$.

Then $\sum_{i=1}^{k+1} (4i+3) = \sum_{i=1}^{k} (4i+3) + 4(k+1)+3$

$= 2k^2+5k+(4k+7)$

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$$= 2k^2+5k+(4k+7)$$

$$= 2k^2+9k+7$$

$$= 2(k^2+2k+1)+5k+5$$

$$= 2(k+1)^2+5(k+1).$$

By mathematical induction, this holds for all $n \in \mathbb{N}$.

<u>Example</u> Show that for $n \in \mathbb{N}$.

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

<u>Proof</u> Induction basis: $\sum_{i=1}^{1} i^3 = 1 = \left(\frac{1(1+1)}{2}\right)^2.$

Induction step: Assume that $\sum_{i=1}^{k} i^3 = \left(\frac{k(k+1)}{2}\right)^2.$

$$= 2k^2 + 9k + 7$$

$$= 2(k^2 + 2k + 1) + 5k + 5$$

$$= 2(k+1)^2 + 5(k+1).$$

By mathematical induction, this holds for all $n \in \mathbb{N}$.

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Then

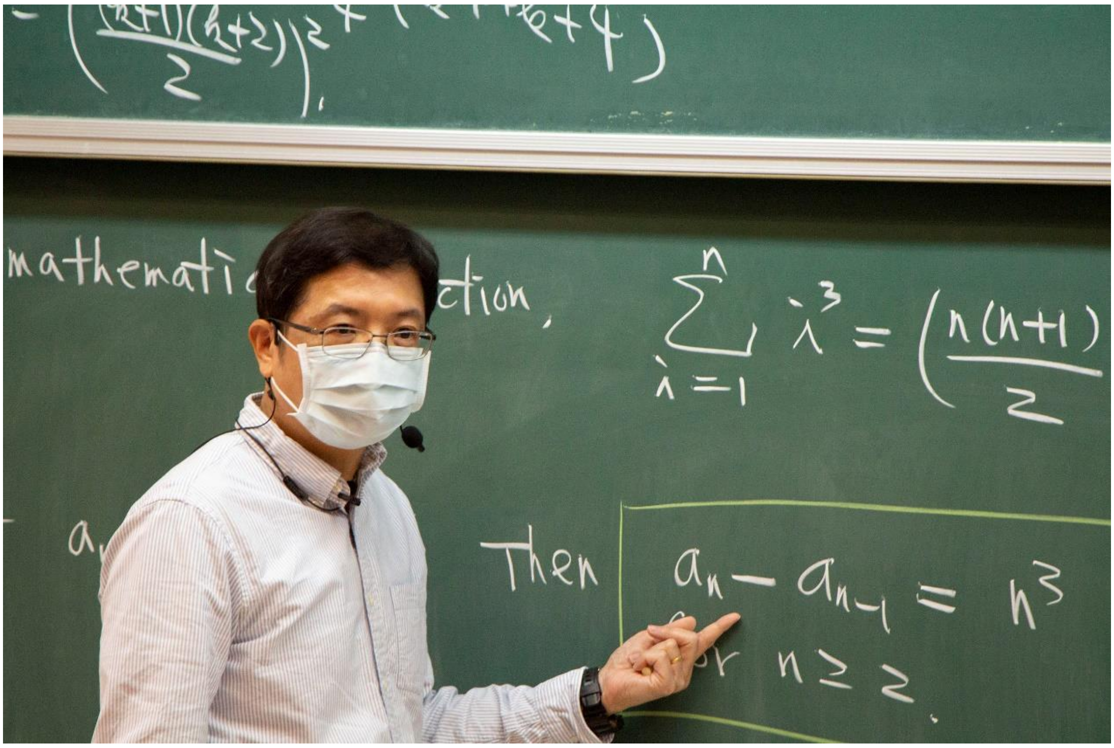
$$\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3$$

$$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

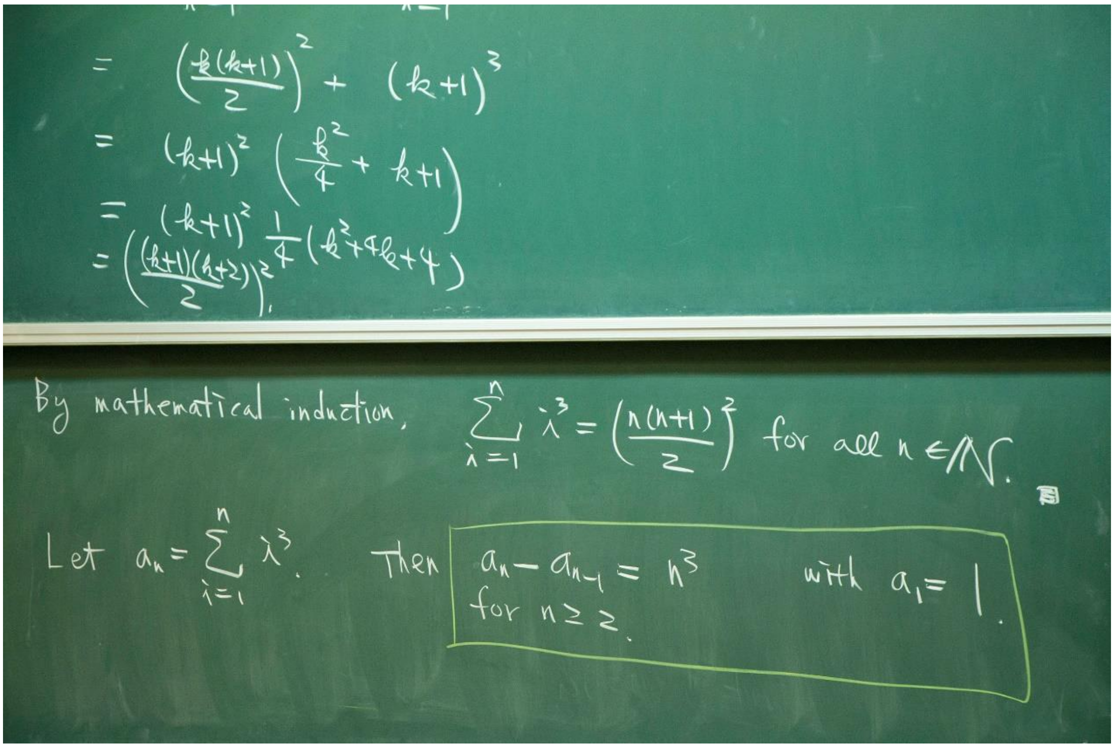
$$= (k+1)^2\left(\frac{k^2}{4} + k + 1\right)$$

$$= (k+1)^2 \frac{1}{4}(k^2 + 4k + 4)$$

$$= \left(\frac{(k+1)(k+2)}{2}\right)^2.$$



mathemati ction,
$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right.$
Then $a_n - a_{n-1} = n^3$
r $n \geq 2$.



$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$
$= (k+1)^2 \left(\frac{k^2}{4} + k+1\right)$
$= (k+1)^2 \frac{1}{4} (k^2+4k+4)$
$= \left(\frac{(k+1)(k+2)}{2}\right)^2.$
By mathematical induction, $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$ for all $n \in \mathbb{N}$. ∎
Let $a_n = \sum_{i=1}^{n} i^3$. Then $a_n - a_{n-1} = n^3$ with $a_1 = 1$.
for $n \geq 2$.

Then $\displaystyle\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3$

$$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

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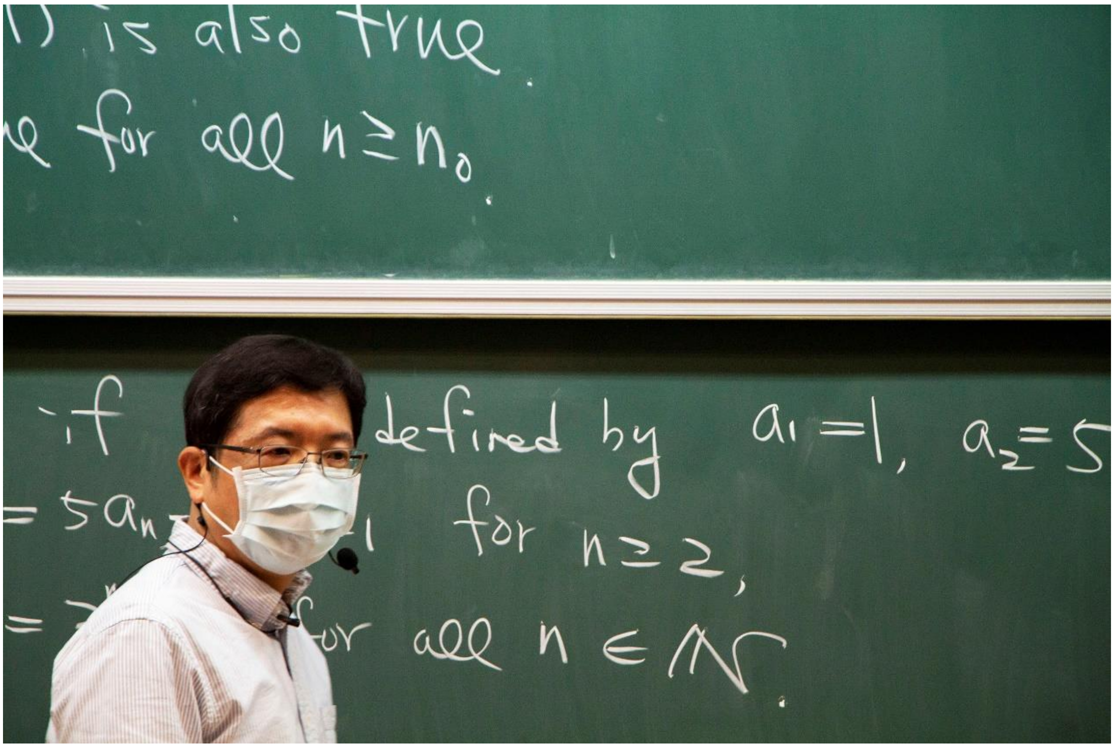
Let $a_n = \displaystyle\sum_{i=1}^{n} i^3$. Then $\boxed{a_n - a_{n-1} = n^3 \text{ for } n \geq 2 \quad \text{with } a_1 = 1.}$

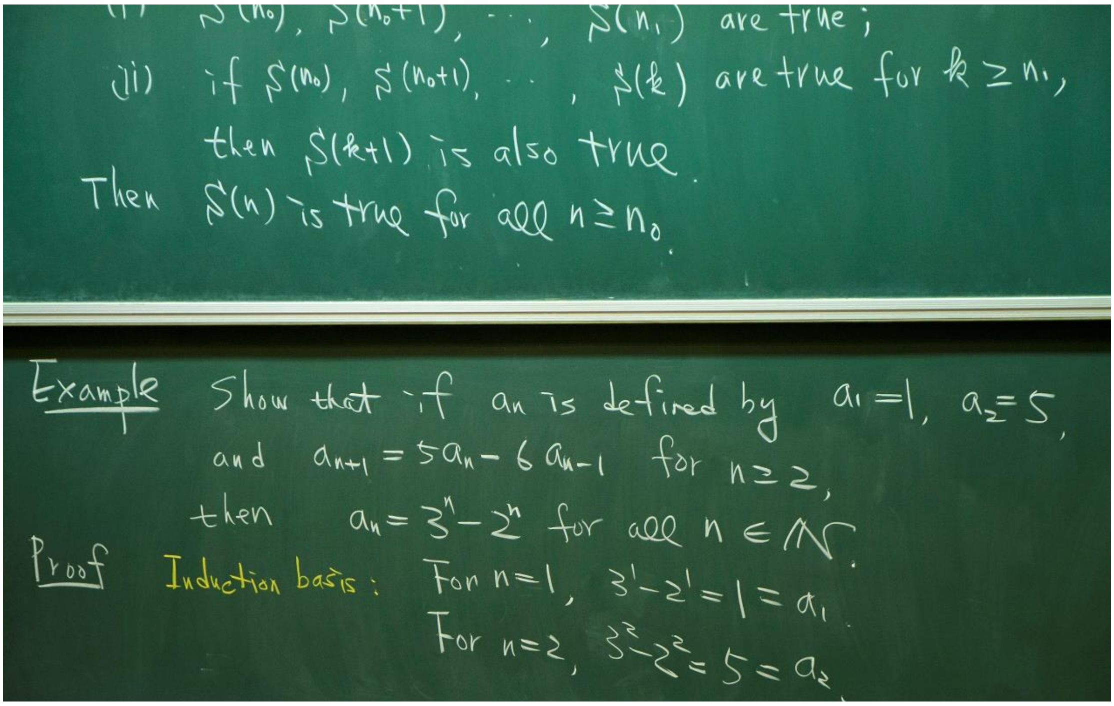
Principle of mathematical induction - Alternative form

(i) $S(n_0), S(n_0+1), \ldots, S(n_1)$ are true;

(ii) if $S(n_0), S(n_0+1), \ldots, S(k)$ are true for $k \geq n_1$, then $S(k+1)$ is also true.

Then $S(n)$ is true for all $n \geq n_0$.


) is also true.
e for all n ≥ n₀.
if defined by a₁ = 1, a₂ = 5
= 5aₙ 1 for n ≥ 2,
= for all n ∈ N.


S(n₀), S(n₀+1), . . . , S(n₁) are true;
(ii) if S(n₀), S(n₀+1), . . . , S(k) are true for k ≥ n₁,
then S(k+1) is also true.
Then S(n) is true for all n ≥ n₀.
Example Show that if aₙ is defined by a₁ = 1, a₂ = 5,
and aₙ₊₁ = 5aₙ − 6aₙ₋₁ for n ≥ 2,
then aₙ = 3ⁿ − 2ⁿ for all n ∈ N.
Proof Induction basis: For n = 1, 3¹ − 2¹ = 1 = a₁.
For n = 2, 3² − 2² = 5 = a₂.

Principle of mathematical induction - Alternative form

(i) $S(n_0), S(n_0+1), \ldots, S(n_1)$ are true;

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Example Show that if a_n is defined by $a_1 = 1$, $a_2 = 5$, and $a_{n+1} = 5a_n - 6a_{n-1}$ for $n \geq 2$, then $a_n = 3^n - 2^n$ for all $n \in \mathbb{N}$.

Proof Induction basis: For $n=1$, $3^1 - 2^1 = 1 = a_1$.

For $n=2$, $3^2 - 2^2 = 5 = a_2$.

Principle of mathematical induction - Alternative form

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Proof Induction basis: For $n=1$, $3^1 - 2^1 = 1 = a_1$.

For $n=2$, $3^2 - 2^2 = 5 = a_2$.

Induction step: Assume that the formula is true for both a_k and a_{k-1}.

Then
$$
\begin{aligned}
a_{k+1} &= 5a_k - 6a_{k-1} \\
&= 5(3^k - 2^k) - 6(3^{k-1} - 2^{k-1}) \\
&= (5\cdot 3 - 6)\,3^{k-1} - (5\cdot 2 - 6)\,2^{k-1} \\
&= 9\cdot 3^{k-1} - 4\cdot 2^{k-1} = 3^{k+1} - 2^{k+1}.
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Therefore, by mathematical induction, $a_n = 3^n - 2^n$ for all $n \in \mathbb{N}$.

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Therefore, by mathematical induction, $a_n = 3^n - 2^n$ for all $n \in \mathbb{N}$.

Example Theorem: Everybody is rich.

Proof Consider $S(n)$: Given any collection of n persons, if at least one of them is rich, then they are all rich.

We prove $S(n)$ by mathematical induction.

Induction basis: $S(1)$ is obviously true.

Induction step: Assume that $S(k)$ is true.

Consider $k+1$ persons $\{\alpha_1, \alpha_2, \cdots, \alpha_k, \alpha_{k+1}\}$

Suppose α_1 is rich. Then $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ is a set of k persons. Since α_1 is rich, by $S(k)$, $\alpha_1, \alpha_2, \cdots, \alpha_k$ are all rich.

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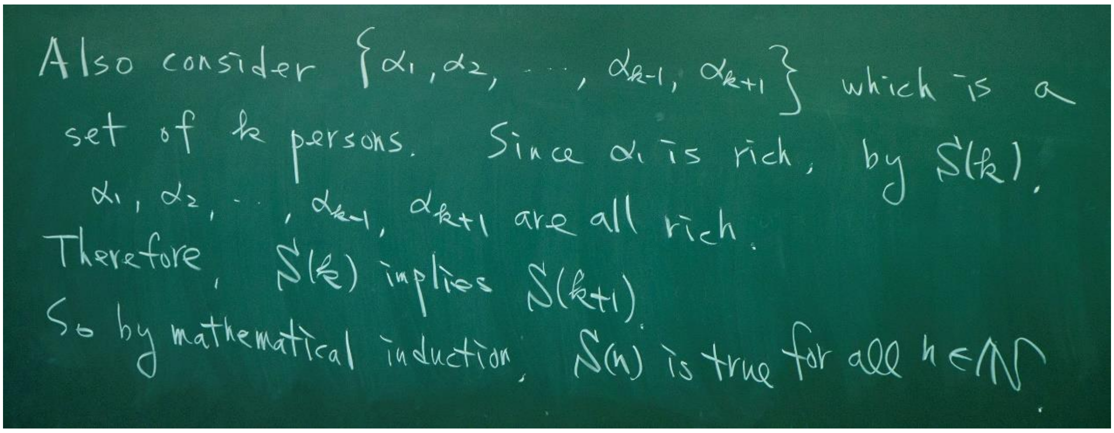
Also consider {α₁, α₂, …, α_{k-1}, α_{k+1}} which is a set of k persons. Since α₁ is rich, by S(k),
α₁, α₂, …, α_{k-1}, α_{k+1} are all rich.
Therefore, S(k) implies S(k+1).
So by mathematical induction, S(n) is true for all n ∈ ℕ.


induction.
true.
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So by mathematical induction, $S(n)$ is true for all $n \in \mathbb{N}$.

Since there certainly exists at least one rich person, it follows that all persons are rich. ∎

<u>Question</u>: What is wrong with the proof?

<u>Answer</u>: The induction step that $S(k)$ implies $S(k+1)$ does not hold for $k=1$

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Functions

Let A and B be nonempty sets.

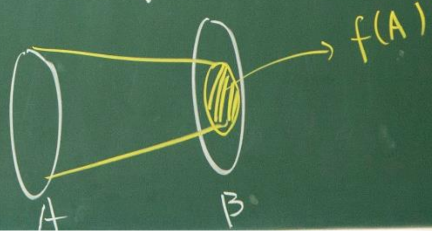
<u>Def</u> A function $f: A \to B$ is a rule that assigns to each element $a \in A$ a unique element $b \in B$.

$f(a) = b$

A: domain of f

B: codomain of f

$f(A) = \{f(a) : a \in A\}$ range of A



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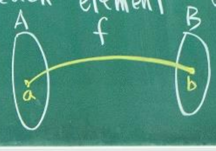

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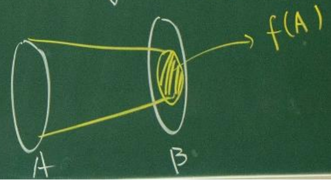

Def A function $f: A \to B$ is called one-to-one, or injective, if $f(a_1) = f(a_2)$ implies $a_1 = a_2$ (or equivalently, $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$).

And f is called an injection.

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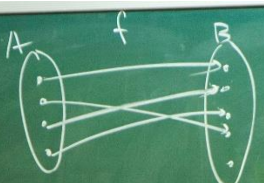

Def A function $f: A \to B$ is called **onto**, or **surjective**, if $f(A) = B$, i.e., for all $b \in B$ there is at least one $a \in A$ with $f(a) = b$. And f is called an **surjection**.

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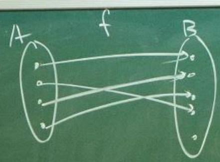

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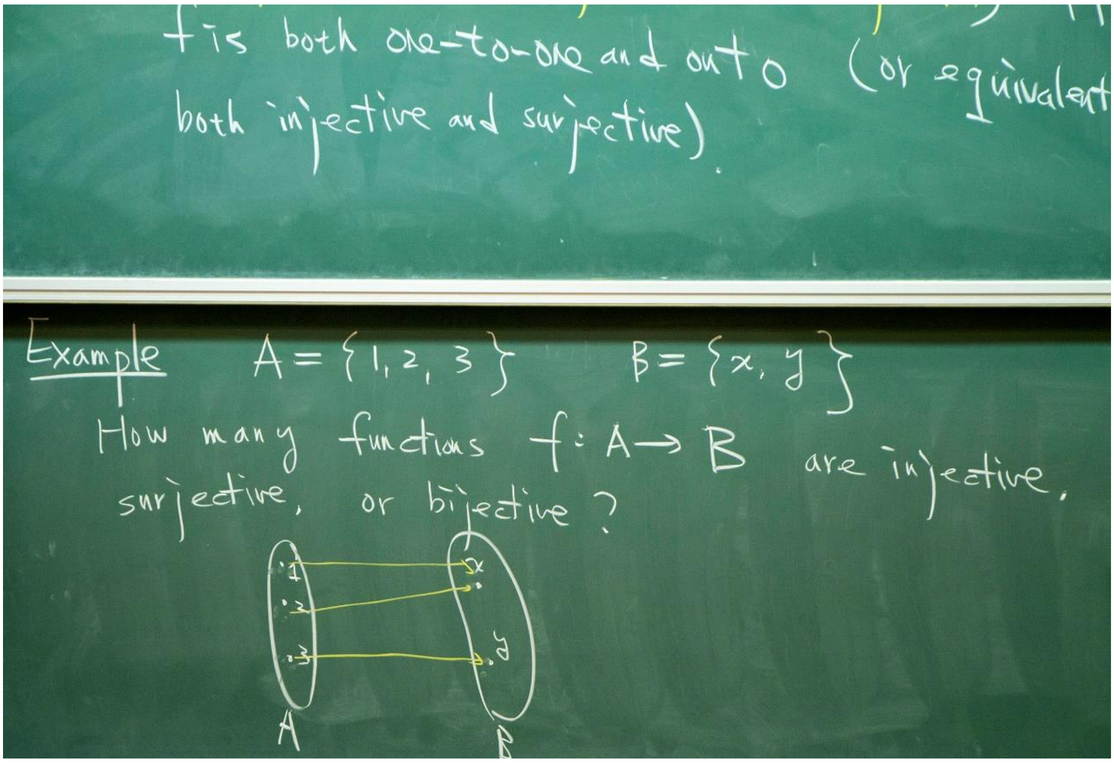
A function $f: A \to B$ is called a **one-to-one correspondence** or **bijective** (a **bijection**) if f is both one-to-one and onto (or equivalently, both injective and surjective).


A = {1, 2, 3}
many functions f: A→
surjective, or bijective ?
1
2
3
x
y
A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both injective and surjective).
Example A = {1, 2, 3} B = {x, y}
How many functions f: A→B are injective,
surjective, or bijective ?
1
2
3
x
y
A

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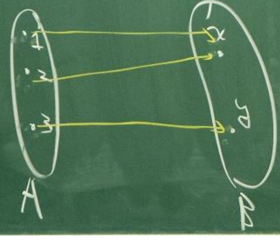
Example $A = \{1, 2, 3\}$ $B = \{x, y\}$

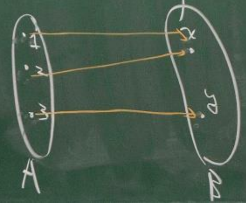
How many functions $f : A \to B$ are injective, surjective, or bijective?

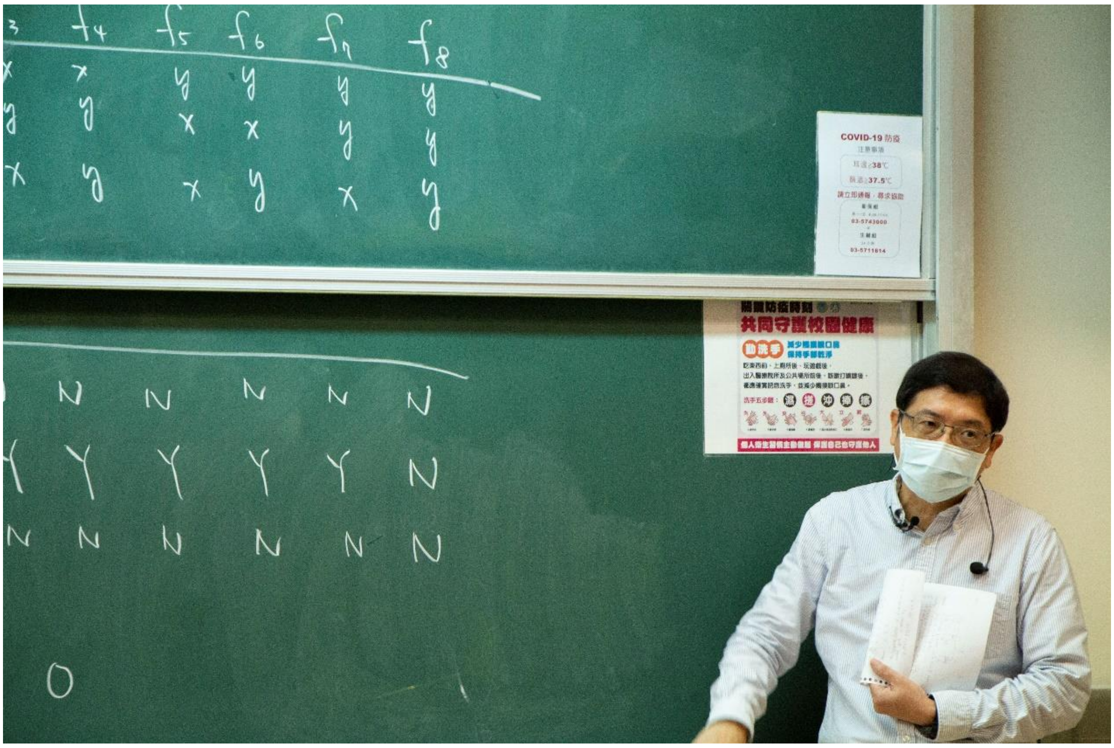


Def A function $f : A \to B$ is called a one-to-one correspondence or bijective (a bijection) if f is both one-to-one and onto (or equivalently, both injective and surjective).

Example $A = \{1, 2, 3\}$ $B = \{x, y\}$

How many functions $f : A \to B$ are injective, surjective, or bijective?

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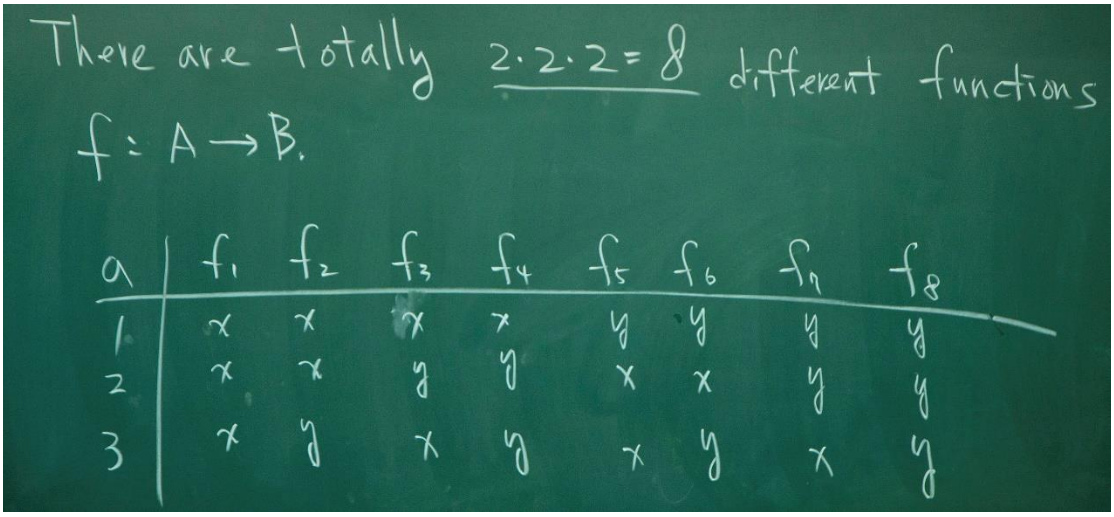
共同守護校園健康


There are totally 2·2·2=8 different functions
f: A→B.
a f1 f2 f3 f4 f5 f6 f7 f8
1 x x x x y y y y
2 x x y y x x y y
3 x y x y x y x y

injective	N	N	N	N	N	N	N	N
surjective	N	Y	Y	Y	Y	Y	Y	N
bijective	N	N	N	N	N	N	N	N

Ans: 0, 6, 0

There are totally $\underline{2 \cdot 2 \cdot 2 = 8}$ different functions $f: A \to B$.

a	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
1	x	x	x	x	y	y	y	y
2	x	x	y	y	x	x	y	y
3	x	y	x	y	x	y	x	y

injective	N	N	N	N	N	N	N	N
surjective	N	Y	Y	Y	Y	Y	Y	N
bijective	N	N	N	N	N	N	N	N

Ans: 0, 6, 0